# Interpolation Parameter and Expansion for a Three-Dimensional Nontrivial Scalar Infrared Fixed Point

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We compute a nontrivial infrared $\phi_3^4$-fixed point by means of an interpolation expansion in fixed dimension. The expansion is formulated for an infinitesimal momentum-space renormalization group. We choose a coordinate representation for the fixed-point interaction in derivative expansion, and compute its coordinates to high orders by means of computer algebra. We compute the series for the critical exponent $v$ up to order 25 of interpolation expansion in this representation, and evaluate it using Padé, Borel–Padé, Borel–conformal–Padé, and $D \log$ –Padé resummation. The resummation returns 0.6262(13) as the value of $v$. Our renormalization group uses canonical resealing, for which $\eta = 0$.

## 1. INTRODUCTION

Nontrivial fixed points are a highly challenging aspect of renormalization theory. Much of what is known about nontrivial fixed point is due to the $\varepsilon$-expansion of Wilson and Fisher [WF72, WK74], which is an interpolation from a critical dimension to the one of interest. In this paper we present another interpolation scheme where the dimension of the underlying (Euclidean) space-time is kept fixed.

A prototype of a nontrivial fixed point is the infrared fixed point of massless $\phi^4$-theory in three dimensions, also called Wilson fixed point

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[WK74]. We choose it as an example for our method. Although it has been investigated by various other means, for instance by Monte Carlo simulations of the three-dimensional Ising model close to criticality, hopping parameter expansion, field theoretic perturbation theory for its scaling limit, and numerical integration of renormalization group flows in a number of setups, our knowledge of it is far from satisfactory. Accurate data is lacking both for its spectrum of critical indices and its anomalous dimension. Its functional form is largely unknown, in particular its locality properties, and its mathematical construction remains an outstanding difficult problem. We mention [ZJ89] and references therein as a guide to the extensive literature. We mention further that all this has been accomplished to a very satisfactory status in the hierarchical approximation by Koch and Wittwer [KW91]. Our interpolation is a brick in the analysis of the full model.

As starting point we choose a functional differential equation from the infinitesimal renormalization group of Wilson [WK74]. Specifically we choose a normal ordered and resealed representation for the fixed point interaction, expressed in terms of a scalar field with nonanomalous scaling dimension. It contains a bilinear renormalization form. This bilinear form is continuously turned on with an auxiliary parameter such that zero gives a linear theory and one restores the full equation. The linear theory is arranged such that the $\phi^4$-interaction acquires the scaling dimension zero in it. We then expand the fixed point interaction into a power series in the interpolation parameter. This part is similar to the $\varepsilon$-expansion. In order to perform the expansion to high orders on the computer, the interaction is written in a basis of interactions which includes a general two point interaction in derivative expansion together with local higher interactions. We compute both the fixed point interaction and the eigenvalue associated with a massive perturbation. The resulting power series are evaluated by means of Padé, $D$ log, Borel-Padé, and Borel-conformal-Padé resummation.

The interpolation idea applies also to other renormalization schemes. The infinitesimal renormalization group is a particularly convenient one because it involves a minimal set of Feynman integrals. Interpolations in fixed dimensions can also be formulated for discrete renormalization group transformations both in continuum regularization and on the lattice, at the expense of dealing with general nonlinear rather than quadratic equations. It is conceivable that our interpolation can be given a meaning beyond perturbation theory.

The paper is organized as follows. In section two we explain the structure of our particular functional differential equation. It is taken from [W96] and is the Wilson equation [WK74] in a kind of interaction

picture. In section three we present our interpolation scheme. It is compared to a naive interpolation which has only a trivial solution. We solve our equations to lowest order to explain their recursive treatment. In section four we discuss their form in a coordinate representation. The result is a set of algebraic recursion relations for the fixed point interaction. They involve a set of structure constants whose computation again involves certain Feynman integrals and multiplicities. We devote section five to this issue. In section six and seven the eigenvalue problem for the scaling fields of the nontrivial fixed point and their anomalous dimensions is treated along the same lines. We restrict our attention to a mass perturbation of the fixed point and the associated critical index $\nu$. In chapter eight and nine the resulting recursions are studied by means of computer algebra.

Our renormalization group uses canonical rescaling. Thereby, the value of the anomalous dimension $\eta$, defined by the decay of the two point function of the field $\phi$ at the fixed point, is built in as $\eta = 0$. We conclude with a brief discussion of this issue. Interpolation expansions for the renormalization group with anomalous resealing will be presented elsewhere.

## 2. RENORMALIZATION GROUP FIXED POINT

We consider a real scalar field $\phi$ on three-dimensional Euclidean space. We use a momentum space renormalization group built from the decomposition of a massless propagator $v$ with exponential ultraviolet regulator. The renormalization group will be formulated in terms of an interaction $V(\phi)$. Concerning the general background, we refer to the work of Wilson [W71], Wilson and Kogut [WK74], and also to Gallavotti [G85]. Our setup will be identical with that in [W96].

We study the nontrivial infrared fixed point in three dimensions as solution to the functional differential equation

$$\left(\mathscr{D}\phi, \frac{\delta}{\delta\phi}\right) V(\phi) = \langle V(\phi) \mid V(\phi)\rangle \tag{1}$$

which was derived in [W96]. Eq. (1) is a normal ordered and rescaled variant of the infinitesimal renormalization group due to Wilson [WK74]. Its origin is a flow equation governing the behaviour of interactions upon the infinitesimal change of a floating cutoff. Eq. (1) gives stationary flows modulo the rescaling of units. The left hand side of (1) is a generator of dilatations

$$\left(\mathscr{D}\phi, \frac{\delta}{\delta\phi}\right) V(\phi) = \frac{d}{dL} V(\phi_L)\Big|_{L=1}, \qquad \phi_L(x) = L^{-1/2}\phi\left(\frac{x}{L}\right) \tag{2}$$

acting on the interaction. The field $\phi$ is here rescaled non-anomalously with its canonical dimension at the trivial fixed point. It should be distinguished from the scaling fields of the infrared fixed point which have nonzero anomalous dimensions. The right hand side of (1) is a bilinear renormalization group form

$$\langle V(\phi) \mid V(\phi) \rangle = \left( \frac{\delta}{\delta\phi_1}, \chi \frac{\delta}{\delta\phi_2} \right) \exp \left\{ \left( \frac{\delta}{\delta\phi_1}, v \frac{\delta}{\delta\phi_2} \right) \right\} V(\phi_1) \left. V(\phi_2) \right|_{\phi_1 = \phi_2 = \phi} \tag{3}$$

It can be visualized as a sum of contractions between two copies of the interaction. Each contraction is made of one *hard* propagator $\chi$ and any number of *soft* propagators $v$. The propagators are here given by

$$\tilde{\chi}(p) = e^{-p^2}, \qquad \tilde{v}(p) = \frac{e^{-p^2}}{p^2} \tag{4}$$

as in [W96]. Eq. (1) is the main dynamical equation in this investigation. Being a differential equation, it has to be supplied with further data to select a particular solution. In a rigorous theory in the sense of Glimm and Jaffe [GJ87], the infrared fixed point should come as a global $Z_2$-symmetric solution, where *global* refers to some criterion of finiteness. Our point of view in this approach will be more modest. An interaction $V(\phi)$ will stand for a power series

$$V(\phi) = \sum_{n=1}^{\infty} \int d^3x_1 \cdots d^3x_{2n} \phi(x_1) \cdots \phi(x_{2n}) \, V_{2n}(x_1,..., x_{2n}) \tag{5}$$

in the field, with symmetric Euclidean invariant distributional kernels given by Fourier integrals

$$V_{2n}(x_1,..., x_{2n}) = \int \frac{d^3p_1}{(2\pi)^3} \cdots \frac{d^3p_{2n}}{(2\pi)^3} e^{i(p_1 x_1 + \cdots + p_{2n} x_{2n})}$$

$$\times (2\pi)^3 \, \delta^{(3)}(p_1 + \cdots + p_{2n}) \, \tilde{V}_{2n}(p_1,..., p_{2n}) \tag{6}$$

of smooth momentum space kernels. I.e., we identify an interaction with its collection of momentum space kernels. The question of convergence of the expansion (5) in powers of fields will not be addressed. It is conceivable that it could be tackled with a suitable norm on the collection of momentum space kernels as a whole.

In the iterative approach to be defined below we will meet at finite order no more than polynomial expressions in the field. We will understand

(1) as a system of differential equations for the momentum space kernels. Its explicit form can be looked up in [W96]. Boundary data is substituted for by the condition of regularity. Homogeneous functions give particular kernels, which correspond to Scaling fields of the trivial fixed point. Expanding a kernel in powers of momentum derivatives, we can always express it in terms of such scaling fields. To distinguish them from the scaling fields of the nontrivial fixed point and also because we will use perturbation theory, we will speak of them as *vertices*.

## 3. INTERPOLATION PARAMETER AND EXPANSION

Our strategy to solve (1) is to interpolate to a solvable situation. If the interpolation is smooth it can be performed by means of perturbation theory. A natural candidate is

$$\left(\mathscr{D}\phi, \frac{\delta}{\delta\phi}\right) V(\phi, z) = z\langle V(\phi, z) \mid V(\phi, z)\rangle \tag{7}$$

with an interpolation parameter $z = 0...1$. It can be thought to turn on continuously the bilinear form, which is identified as the source of troubles. The interpolation (7) is inappropriate for the following reason, when the dimension parameter is fixed to three. Expand the interpolated interaction as a function of the interpolation parameter in a power series

$$V(\phi, z) = \sum_{r=0}^{\infty} z^r V^{(r)}(\phi) \tag{8}$$

Unfortunately there is little hope that (8) has a finite radius of convergence both in the case of (7) and the interpolation (15) considered below. To be cautious we will therefore view (8) as a formal power series and interpret all equations below in this sense. It will however be argued that nonperturbative information can be extracted by Borel resummation. In order to solve (7), the expansion (8) has to satisfy

$$\left(\mathscr{D}\phi, \frac{\delta}{\delta\phi}\right) V^{(r)}(\phi) = \sum_{s=0}^{r-1} = 0\langle V^{(s)}(\phi) \mid V^{(r-1-s)}(\phi)\rangle \tag{9}$$

to every order $r \in \mathbb{N}$, with the understanding $V^{(-1)}(\phi) = 0$. In particular it requires the interaction to satisfy

$$\left(\mathscr{D}\phi, \frac{\delta}{\delta\phi}\right) V^{(0)}(\phi) = 0 \tag{10}$$

to zeroth order. In other words, the zeroth order has to be a marginal scaling field of the trivial fixed point. In three dimensions we have two marginal scaling fields, a wave function term and a $\phi^6$-vertex, to be abbreviated as

$$\mathcal{O}_{1,1}(\phi) = \int d^3x \, \phi(x)(-\Delta) \, \phi(x), \qquad \mathcal{O}_{3,0}(\phi) = \int d^3x \, \phi(x)^6 \qquad (11)$$

We emphasize that vertices should be understood as momentum space kernels at zero momentum and their Taylor expansions. Each of them comes with a formal orthogonal projector $\mathcal{P}_{1,1}$ and $\mathcal{P}_{3,0}$, selecting the corresponding vertex from a general interaction (5). The zeroth order has to be a linear combination

$$V^{(0)}(\phi) = V_{1,1}^{(0)} \mathcal{O}_{1,1}(\phi) + V_{3,0}^{(0)} \mathcal{O}_{3,0}(\phi) \qquad (12)$$

The coupling constants are not determined by the zeroth order Eq. (10). To first order, (9) reads

$$\left(\mathcal{D}\phi, \frac{\delta}{\delta\phi}\right) V^{(1)}(\phi) = \langle \, V^{(0)}(\phi) \mid V^{(0)}(\phi) \rangle \qquad (13)$$

But Eqs. (12) and (13) together have only a trivial solution. The left hand side of (13) cannot contain the vertices (11) because the dilatation generator $(\mathcal{D}\phi, \delta/\delta\phi)$ has no marginal image. Therefore, (13) requires that

$$\mathcal{P}_{1,1} \langle \, V^{(0)}(\phi) \mid V^{(0)}(\phi) \rangle = \mathcal{P}_{3,0} \langle \, V^{(0)}(\phi) \mid V^{(0)}(\phi) \rangle = 0 \qquad (14)$$

Computing the bilinear form with two copies of (12) inevitably gives $V_{1,1}^{(0)} = V_{3,0}^{(0)} = 0$. Eq. (7) is thus an inappropriate interpolation and has to be given up.

A way around the obstacle is to interpolate simultaneously the dimensionality of the theory. This is the strategy of the $\varepsilon$-expansion of Wilson and Fisher [WF72] in a field theoretic setup. Another way is to interpolate the scaling dimension, remaining firmly in three dimensions. We choose this second route and replace (7) by

$$\left[\left(\mathcal{D}\phi, \frac{\delta}{\delta\phi}\right) - 1\right] V(\phi, z) = z \langle \, V(\phi, z) \mid V(\phi, z) \rangle - z V(\phi, z) \qquad (15)$$

The power series expansion (8) solves (15) if the coefficients satisfy the system of differential equations

$$\left[\left(\mathcal{D}\phi, \frac{\delta}{\delta\phi}\right) - 1\right] V^{(r)}(\phi) = \sum_{s=0}^{r-1} \langle V^{(s)}(\phi) \mid V^{(r-1-s)}(\phi)\rangle - V^{(r-1)}(\phi) \qquad (16)$$

to all orders $r \in \mathbf{N}$. To order zero, (16) requires now

$$\left[\left(\mathcal{D}\phi, \frac{\delta}{\delta\phi}\right) - 1\right] V^{(0)}(\phi) = 0 \qquad (17)$$

in contrast to (10). The zeroth order interaction is now a scaling field with unit scaling dimension. In three dimensions we have only one candidate, the $\phi^4$-vertex

$$\mathcal{O}_{2,0}(\phi) = \int d^3x \, \phi(x)^4 \qquad (18)$$

The zeroth order interaction thus has to be proportional to (18). The proportionality factor is the $\phi^4$-coupling. It is not determined by the zeroth order Eq. (17). We conclude that

$$V^{(0)}(\phi) = V_{2,0}^{(0)} \mathcal{O}_{2,0}(\phi) \qquad (19)$$

This expansion proves to have indeed a nontrivial solution. To see this, consider the first order equation in (16). It reads

$$\left[\left(\mathcal{D}\phi, \frac{\delta}{\delta\phi}\right) - 1\right] V^{(1)}(\phi) = \langle V^{(0)}(\phi) \mid V^{(0)}(\phi)\rangle - V^{(0)}(\phi) \qquad (20)$$

Eq. (20) cannot have a $\phi^4$-vertex on its left hand side. Therefore it is required that

$$\mathcal{P}_{2,0}\{\langle V^{(0)}(\phi) \mid V^{(0)}(\phi)\rangle - V^{(0)}(\phi)\} = 0 \qquad (21)$$

Computing the bilinear form, this condition reads explicitly

$$V_{2,0}^{(0)}\{144\tilde{\chi} * \tilde{v}(0) \, V_{2,0}^{(0)} - 1\} = 0 \qquad (22)$$

where $*$ means convolution times $(2\pi)^{-3}$. Besides the trivial solution $V_{2,0}^{(0)} = 0$ it has a nontrivial solution

$$V_{2,0}^{(0)} = \frac{1}{144\tilde{\chi} * \tilde{v}(0)} = \frac{(2\pi)^{3/2}}{72} = 0.21874445\dots \qquad (23)$$

The value of the $\phi^4$-coupling at any given order will in fact be determined by the equations at the next order, a feature of this particular interpolation expansion. To first order, the interaction can be split into

$$V^{(1)}(\phi) = V_{2,0}^{(1)} \mathcal{O}_{2,0}(\phi) + \mathscr{P}_{2,0}^{\perp} V^{(1)}(\phi) \tag{24}$$

with $\mathscr{P}_{2,0}^{\perp} = 1 - \mathscr{P}_{2,0}$ the projector on the formal orthogonal complement. Eq. (20) defines a system of first order differential equations for the momentum space kernels therein. They have a unique integral in the space of smooth functions of momenta, see [W96]. We denote this integral by

$$\mathscr{P}_{2,0}^{\perp} V^{(1)}(\phi) = \left[ \left( \mathscr{D}\phi, \frac{\delta}{\delta\phi} \right) - 1 \right]^{-1} \mathscr{P}_{2,0}^{\perp} \langle V^{(0)}(\phi) \mid V^{(0)}(\phi) \rangle \tag{25}$$

This iterative scheme carries on to every order of interpolation expansion. Consider Eq. (16) at order $r \geq 2$. The first step of the iteration is to compute $V_{2,0}^{(r-1)}$ at order $r-1$. Making use of (23), its value follows from

$$V_{2,0}^{(r-1)} \mathcal{O}_{2,0}(\phi) = -2 \mathscr{P}_{2,0} \langle V^{(0)}(\phi) \mid \mathscr{P}_{2,0}^{\perp} V^{(r-1)}(\phi) \rangle$$

$$- \sum_{s=1}^{r-2} \mathscr{P}_{2,0} \langle V^{(s)}(\phi) \mid V^{(r-1-s)}(\phi) \rangle \tag{26}$$

Again one splits the order $r$ interaction into

$$V^{(r)}(\phi) = V_{2,0}^{(r)} \mathcal{O}_{2,0}(\phi) + \mathscr{P}_{2,0}^{\perp} V^{(r)}(\phi) \tag{27}$$

and computes the formal orthogonal complement to the $\phi^4$-vertex by integrating the first order differential Eqs. (16). The result can be written as

$$\mathscr{P}_{2,0}^{\perp} V^{(r)}(\phi) = \left[ \left( \mathscr{D}\phi, \frac{\delta}{\delta\phi} \right) - 1 \right]^{-1}$$

$$\times \left\{ \sum_{s=0}^{r-1} \mathscr{P}_{2,0}^{\perp} \langle V^{(s)}(\phi) \mid V^{(r-1-s)}(\phi) \rangle - \mathscr{P}_{2,0}^{\perp} V^{(r-1)}(\phi) \right\} \tag{28}$$

Thereafter it is time to proceed to the equations at order $r+1$. For this scheme to work as above it is important that the kernel of $(\mathscr{D}\phi, \delta/\delta\phi) - 1$ be one-dimensional. Otherwise we would have to compute further order $r-1$ data from the equations to order $r$. An example where this happens is the $\phi^4$-trajectory in four dimensions [W96].

Although being in principle doable, the computation of this scheme to very high orders of interpolation expansion is a tedious enterprise. The

main work is the computation of a wealth of Feynman kernels generated in the course of iterations A lower order analysis of this program will be presented elsewhere. In this paper we choose to evaluate the expansion to high orders for a sub-class of contributions in the iteration. For this purpose we reformulate the fixed point Eq. (15) into an algebraic system of equations for a set of coupling constants. We find it interesting by its own. It also allows to perform the expansion on a computer.

## 4. COORDINATE REPRESENTATION

We choose a system of vertices $\mathcal{O}_i(\phi)$ labelled by elements $i$ of an index set $\mathcal{I}$. The vertices will be required to be $Z_2$-symmetric, Euclidean invariant, and linearly independent. They will also be required to be regular in the sense that they are given by smooth momentum space kernels. We choose the system such that the dilatation generator acts linearly on it through a scaling dimension matrix

$$\left(\mathcal{D}\phi, \frac{\delta}{\delta\phi}\right) \mathcal{O}_i(\phi) = \sum_{j \in \mathcal{I}} \mathcal{O}_j(\phi) \, \Sigma_i^j \tag{29}$$

We restrict our attention to systems with the property that the scaling dimension matrix is diagonalizable. Non diagonalizable matrices will not be considered here. In this case, we can arrange the system to consist of eigenvectors

$$\left(\mathcal{D}\phi, \frac{\delta}{\delta\phi}\right) \mathcal{O}_i(\phi) = \sigma_i \mathcal{O}_i(\phi) \tag{30}$$

In other words, we take $\mathcal{O}_i(\phi)$ to be a scaling field of the trivial fixed point with scaling dimension $\sigma_i$. Recall that such vertices are given by homogeneous momentum space kernels. Eq. (29) says that the system closes under the action of an infinitesimal dilatation. We also require it to close under the action of the bilinear renormalization group form. For any two vertices $\mathcal{O}_i(\phi)$ and $\mathcal{O}_j(\phi)$ the bilinear form (3) will be assumed to be a linear combination

$$\langle \mathcal{O}_i(\phi) \mid \mathcal{O}_j(\phi) \rangle = \sum_{k \in \mathcal{I}} \mathcal{O}_k(\phi) \, F_{i,j}^k \tag{31}$$

with a set of structure constants $F_{i,j}^k$. The scaling dimensions $\sigma_i$ and the structure constants $F_{i,j}^k$ comprise all the information needed in the

following about the system of vertices. We remark that the structure constants are well defined through

$$F^k_{i,j} \mathcal{O}_k(\phi) = \mathscr{P}_k \langle \mathcal{O}_i(\phi) \mid \mathcal{O}_j(\phi) \rangle \tag{32}$$

even when the system does not close under the bilinear form. In this case (31) holds only up to an error term. Below we will indeed work with an approximation of this kind and argue that the error term is small.

We define a coordinate representation for the interpolated interaction in terms of a given system of vertices as

$$V(\phi, z) = \sum_{i \in \mathscr{I}} \mathcal{O}_i(\phi) \, V^i(z) \tag{33}$$

The idea is then to investigate the interpolation (15) for the infrared fixed point by means of the parameter dependent coordinates (33). Eq. (15) becomes a system of algebraic equations

$$(\sigma_k - 1) \, V^k(z) = z \sum_{i,j \in \mathscr{I}} F^k_{i,j} V^i(z) \, V^j(z) - z V^k(z) \tag{34}$$

in the coordinate representation. The advantage of (34) as compared to (15) is that we are no longer dealing with differential equations for momentum space kernels. Their integration is hidden in the structure constants. If the interpolation is smooth, we can expand the coordinate functions into power series

$$V^k(z) = \sum_{r=0}^{\infty} z^r V^{k,\,r} \tag{35}$$

By standard arguments (35) is expected to be singular but Borel summable. Our below evaluation of (35) supports this expectation. Eq. (35) yields a solution to (34) in the sense of a formal power series in $z$ if the coefficients obey

$$(\sigma_k - 1) \, V^{k,\,r} = \sum_{s=0}^{r-1} \sum_{i,j \in \mathscr{I}} F^k_{i,j} V^{i,\,s} V^{j,\,r-1-s} - V^{k,\,r-1} \tag{36}$$

holds for all couplings $k \in \mathscr{I}$ to all orders $r \in \mathbf{N}$ of interpolation expansion. We organize (36) into a recursion relation which can be solved on the computer. To zeroth order (36) simplifies to the linear equation

$$(\sigma_k - 1) \, V^{k,\,0} = 0 \tag{37}$$

We assume that our system of vertices contains only one element labelled by $k = \underline{2} = (2, 0)$ such that $\sigma_2 = 1$. This element is of course the $\phi^4$-vertex (18). All other elements are assumed to have scaling dimensions different from one. Then (37) implies that

$$V^{k, 0} = V^{2, 0} \delta_{\underline{2}, k} \tag{38}$$

The value of $V^{2, 0}$ is as above determined by (36) to order one,

$$(\sigma_k - 1) V^{k, 1} = V^{2, 0}(F^2_{\underline{2}, \underline{2}} V^{2, 0} - \delta_{2, k}) \tag{39}$$

Evaluating (39) for $k = \underline{2}$ it follows immediately that we have

$$V^{2, 0} = \frac{1}{F^2_{\underline{2}, \underline{2}}} \tag{40}$$

besides the trivial solution $V^{2, 0} = 0$. Eq. (39) does not tell the value of $V^{2, 1}$. But for $k \in \mathcal{S} \setminus \{\underline{2}\}$ it gives

$$V^{k, 1} = \frac{F^k_{\underline{2}, \underline{2}}(V^{2, 0})^2}{\sigma_k - 1} \tag{41}$$

Eq. (40) and (41) are of course the coordinate versions of (23) and (25). The strategy to any order $r > 1$ is again to first compute $V^{2, r-1}$ and thereafter $V^{k, r}$ for $k \in \mathcal{S} \setminus \{\underline{2}\}$. The explicit formulas are

$$V^{2, r-1} = -2 \sum_{i \in \mathcal{S} \setminus \{\underline{2}\}} F^2_{i, \underline{2}} V^{2, 0} V^{i, r-1} - \sum_{s=1}^{r-2} \sum_{i, j \in \mathcal{S}} F^2_{i, j} V^{i, s} V^{j, r-1-s} \tag{42}$$

and

$$V^{k, r} = \frac{1}{\sigma_k - 1} \left\{ \sum_{s=0}^{r-1} \sum_{i, j \in \mathcal{S}} F^k_{i, j} V^{i, s} V^{j, r-1-s} - V^{k, r-1} \right\} \tag{43}$$

in complete analogy to (26) and (28). Thus once we know the scaling dimensions and the Structure constants, the iteration proceeds by means of purely algebraic operations. We remark that the sums in (42) and (43) will be finite in the system of vertices considered below. The reason is that the outcome of the bilinear form of two monomials in the field is a polynomial in the field of finite order, and consists only of connected vertices. A very interesting question is whether it is possible to find finite systems of vertices that close under both (29) and (31). It is clear that this cannot be achieved in terms of polynomial vertices. Unfortunately no such system is known in three dimensions.

## 5. STRUCTURE CONSTANTS

We consider the following system of vertices. First we include a full two point vertex in derivative expansion. A convenient notation for it is

$$\mathcal{O}_{1,\alpha}(\phi) = \int d^3x \, \phi(x)(-\Delta)^\alpha \phi(x) \tag{44}$$

where $\alpha = 0, 1, 2,...$. Second we include local $(2n)$-point vertices with arbitrary many external legs. They will be abbreviated as

$$\mathcal{O}_{n,0}(\phi) = \int d^3x \, \phi(x)^{2n} \tag{45}$$

where $n = 2, 3, 4,...$. Notice that both (44) and (45) meet the demands stated at the beginning of the previous section. More general interactions include also momentum dependent higher vertices (45). They will not be considered here. our index set is thus

$$\mathscr{I} = \{1\} \times \{\alpha \in \mathbf{N} \mid \alpha \geqslant 0\} \cup \{n \in \mathbf{N} \mid n \geqslant 2\} \times \{0\} \tag{46}$$

and $\underline{2} = (2, 0)$. The bilinear form does *not* close under this set of vertices. For instance two local vertices (45) contract in general to a bilocal vertex. Thus if we perform an iteration (42) and (43) with this system of vertices, we make a systematic error due to the truncation of the system. Our ansatz rests upon the assumption that nonlocal higher vertices are small compared to their local parts.

The scaling dimensions of (44) and (45) come out as

$$\sigma_{1,\alpha} = 2 - 2\alpha, \qquad \sigma_{n,0} = 3 - n \tag{47}$$

The structure constants for this set of vertices come out as follows. Two quadratic vertices always contract again to a quadratic vertex. The associated structure constants are computed to

$$F^{(1,\alpha)}_{(1,\beta),(1,\gamma)} = 4 \frac{(-1)^{\alpha-\beta-\gamma}}{(\alpha-\beta-\gamma)!} \Theta_{\alpha,\beta+\gamma} \tag{48}$$

where $\Theta_{a,b} = 1$ for $a \geqslant b$ and zero else. A quadratic vertex and a higher vertex return upon pairing both a quadratic vertex and a higher vertex. First we have

$$F^{(1,\gamma)}_{(1,\alpha),(2,0)} = 24\tilde{K}_{1,\alpha}(0) \, \delta_{\gamma,0} \tag{49}$$

The structure constant (49) involves the one loop integral

$$\tilde{K}_{1,\alpha}(p) = \int \frac{d^3q}{(2\pi)^3} \, \tilde{v}(q)(q^2)^{\alpha} \, \tilde{\chi}(p-q) \tag{50}$$

at zero momentum. Recall that the propagators are given by (4). The exponential regulator gives a convergent integral which is evaluated in (58). Second we have a one loop contribution

$$F^{(m-1,0)}_{(1,\alpha),(m,0)} = 4m(2m-1) \, \tilde{K}_{1,\alpha}(0) \tag{51}$$

as well as a zero loop contribution

$$F^{(m,0)}_{(1,\alpha),(m,0)} = 4m \, \delta_{\alpha,0} \tag{52}$$

This last pairing also contributes to momentum dependent higher vertices which we neglect. Two higher vertices yield upon pairing both a quadratic vertex and higher vertices. One quadratic term is

$$F^{(1,\gamma)}_{(n,0),(n,0)} = 2n(2n-1)(2n)! \, \tilde{K}^{(\gamma)}_{2n-2}(0) \tag{53}$$

with the $(2n-2)$-loop (the number of soft propagators $v$) integral

$$\tilde{K}_{2n-2}(p) = \tilde{v} * \cdots * \tilde{v} * \tilde{\chi}(p) \tag{54}$$

expanded into

$$\tilde{K}_{2n-2}(p) = \sum_{\alpha=0}^{\infty} (p^2)^{\alpha} \, \tilde{K}^{\alpha}_{2n-2}(0) \tag{55}$$

at zero momentum. A second quadratic term is

$$F^{(1,\gamma)}_{(n,0),(n-1,0)} = (n-1)(2n)! \, \tilde{K}_{2n-2}(0) \, \delta_{\gamma,0} \tag{56}$$

This second term is exactly local. The general higher vertex content of the pairing of two higher vertices is summarized in

$$F^{(l,0)}_{(n,0),(m,0)} = \frac{(2n)! \, (2m)!}{(n+m-l-1)! \, (m+l-n)! \, (l+n-m)!}$$
$$\times \tilde{K}_{n+m-l-1}(0) \, \Theta_{n+m,l+1} \Theta_{m+l,n} \Theta_{l+n,m} \tag{57}$$

This last set of structure constants (57) alone defines a local approximation for the renormalization group fixed point. As mentioned above, the general outcome of the pairing of two higher vertices also contains momentum dependent vertices which are not encorporated in (57). All other structure constants between vertices in $\mathcal{S}$ are zero.

The one loop integral (50) is evaluated to

$$\tilde{K}_{1,\alpha}(0) = \frac{1}{8\pi^2} 2^{1/2-\alpha} \Gamma(\alpha + 1/2) \tag{58}$$

The $l$-loop integral (54) is computed as a function of the external momentum squared to

$$\tilde{K}_l(p) = (4\pi)^{-3l/2} \int_1^\infty d\alpha_1 \cdots d\alpha_l A^{-3/2} \exp\left(\frac{-B}{A} p^2\right) \tag{59}$$

with the abbreviations

$$A = \sum_{m=1}^{l+1} \prod_{n \neq m} \alpha_n, \qquad B = \prod_{m=1}^{l} \alpha_m \tag{60}$$

where $\alpha_{l+1} = 1$. Its momentum derivatives at zero can be reduced further to a one-dimensional integral

$$\tilde{K}_l^{(\alpha)}(0) = \frac{1}{(8\pi)^l} \frac{(-1)^\alpha}{\alpha! \, \Gamma(\alpha + 3/2)} \int_0^\infty dx \, x^{\alpha + 1/2} e^{-x} \left\{ \frac{\mathrm{erf}(\sqrt{x})}{\sqrt{x}} \right\}^l \tag{61}$$

This remaining integral can be done explicitly at least in the one-loop case. We evaluated it in the general case numerically to high accuracy (45 digits) on the computer.

## 6. EIGENVALUE PROBLEM FOR CRITICAL INDICES

The fixed point Eq. (1) comes together with an eigenvalue problem

$$\left[ \left( \mathcal{D}\phi, \frac{\delta}{\delta\phi} \right) - \lambda \right] W(\phi) = 2\langle V(\phi) \mid W(\phi) \rangle \tag{62}$$

defining scaling fields $W(\phi)$ and their anomalous dimensions $\lambda$. We emphasize that $W(\phi)$ is a composite field of $\phi$. The spectrum of anomalous dimensions is the object of principle interest associated with a fixed point. It directly determines the critical exponents, see Wilson and Kogut [WK74].

The interpolation (15) is accompanied by an interpolation of (62), given by

$$\left[ \left( \mathcal{D}\phi, \frac{\delta}{\delta\phi} \right) - \lambda(z) \right] W(\phi, z) = 2z\langle V(\phi, z) \mid W(\phi, z) \rangle \tag{63}$$

Eq. (63) can be solved by means of perturbation theory. We expand not only the interaction (8), but also the scaling field and its anomalous dimension into power series in the interpolation parameter,

$$W(\phi, z) = \sum_{r=0}^{\infty} z^r W^{(r)}(\phi), \qquad \lambda(z) = \sum_{r=0}^{\infty} z^r \lambda^{(r)} \tag{64}$$

We interpret (64) in the sense of a formal power series. It yields a solution to (63) if the coefficients satisfy the system of differential equations

$$\left( \mathscr{D}\phi, \frac{\delta}{\delta\phi} \right) W^{(r)}(\phi) - \sum_{s=0}^{r} \lambda^{(s)} W^{(r-s)}(\phi) = 2 \sum_{s=0}^{r-1} \langle V^{(s)}(\phi) \mid W^{(r-1-s)}(\phi) \rangle \tag{65}$$

This system can be integrated iteratively. To order zero, (65) becomes the eigenvalue problem

$$\left[ \left( \mathscr{D}\phi, \frac{\delta}{\delta\phi} \right) - \lambda^{(0)} \right] W^{(0)}(\phi) = 0 \tag{66}$$

The zeroth order $W^{(0)}(\phi)$ thus has to be a scaling field of the trivial fixed point, and $\lambda^{(0)}$ has to be its scaling dimension. With each scaling field of the trivial fixed point is therefore associated in perturbation theory a scaling field of the nontrivial fixed point.

Let us consider for definiteness the perturbation associated with a mass term

$$W^{(0)}(\phi) = \mathcal{O}_{1,0}(\phi), \qquad \mathcal{O}_{1,0}(\phi) = \int d^3x \, \phi(x)^2 \tag{67}$$

Then the zeroth order eigenvalue is of course $\lambda^{(0)} = \sigma_{1,0} = 2$. As a perturbation of the nontrivial fixed point, (67) turns out to be relevant. The associated nontrivial renormalized trajectory in the sense of [W96] describes the renormalization group flow of a nontrivial massive field theory. Associated with it is the critical exponent

$$\nu = \frac{1}{\lambda} \tag{68}$$

The mass perturbation (67) is nondegenerate in the sense that the kernel of $(\mathscr{D}\phi, \delta/\delta\phi) - 2$ is one-dimensional. The formal orthogonal projector on this one-dimensional kernel is $\mathscr{P}_{1,0}$. Another nondegenerate perturbation is the scaling field associated with $\mathcal{O}_{2,0}(\phi)$. The ones associated with $\mathcal{O}_{1,1}(\phi)$

and $\mathcal{O}_{3,0}(\phi)$ on the other hand form a degenerate marginal duplet. We will restrict our attention to the nondegenerate case for the sake of notational economy. The kernel of $(\mathcal{D}\phi, \delta/\delta\phi) - \lambda^{(0)}$ will thus be assumed to be one-dimensional. The formal orthogonal projector on this rank one kernel will be denoted by $\mathcal{P}$. To first order (65) becomes the differential equation

$$\left[\left(\mathcal{D}\phi, \frac{\delta}{\delta\phi}\right) - \lambda^{(0)}\right] W^{(1)}(\phi) - \lambda^{(1)} W^{(0)}(\phi) = 2\langle V^{(0)}(\phi) \mid W^{(0)}(\phi)\rangle \quad (69)$$

The first order correction to the eigenvalue follows from (69) by projection with $\mathcal{P}$. We have that

$$\lambda^{(1)} W^{(0)}(\phi) = -2\mathcal{P}\langle V^{(0)}(\phi) \mid W^{(0)}(\phi)\rangle \quad (70)$$

Spelled out explicitly for the mass perturbation (67), Eq. (70) says that

$$\lambda^{(1)} = -48\tilde{\chi} * \tilde{v}(0) \, V_{2,0}^{(0)} = \frac{-1}{3} \quad (71)$$

It is amusing that this first order correction can be inferred without having to compute the convolution integral, because the convolution integral in (71) is canceled exactly by the one in (23). Next we impose the normalization condition that

$$\mathcal{P} W^{(1)}(\phi) = 0 \quad (72)$$

This condition is appropriate in the nondegenerate case because (70) already takes care of the $\mathcal{P}$-information contained in (69). The orthogonal complement is then integrated as in the case of the interaction. The outcome is

$$\mathcal{P}^\perp W^{(1)}(\phi) = 2\left[\left(\mathcal{D}\phi, \frac{\delta}{\delta\phi}\right) - \lambda^{(0)}\right]^{-1} \mathcal{P}^\perp \langle V^{(0)}(\phi) \mid W^{(0)}(\phi)\rangle \quad (73)$$

This scheme carries on immediately to every order of interpolation expansion. Projecting (65) to $\mathcal{P}$, we first deduce that

$$\lambda^{(r)} W^{(0)}(\phi) = 2\sum_{s=0}^{r-1} \mathcal{P}\langle V^{(s)}(\phi) \mid W^{(r-1-s)}(\phi)\rangle \quad (74)$$

This equation determines the order $r$ eigenvalue in terms of lower order data. Generalizing (72), we impose the normalization condition

$$\mathcal{P} W^{(r)}(\phi) = 0 \quad (75)$$

for $r \geq 1$. Then to order $r$ we are left with the computation of

$$
\mathscr{P}^{\perp} W^{(r)}(\phi) = \left[ \left( \mathscr{D}\phi, \frac{\delta}{\delta\phi} \right) - \lambda^{(0)} \right]^{-1}
$$

$$
\times \left\{ \sum_{s=1}^{r-1} \lambda^{(s)} \mathscr{P}^{\perp} W^{(r-1)}(\phi) + 2 \sum_{s=0}^{r-1} \mathscr{P}^{\perp} \langle V^{(s)}(\phi) \mid W^{(r-1-s)}(\phi) \rangle \right\}
$$

$$(76)$$

This scheme iterates to every order of interpolation expansion. Recall that the inverse of the dilatation generator in (76) involves the integration of a first order partial differential equation. As in the case of the fixed point, the explicit computation of this program to very high orders requires considerable computational resources. In this paper we restrict our attention to a partial resummation by means of our coordinate representation.

## 7. EIGENVALUE PROBLEM IN COORDINATES

In this section we perform the interpolation expansion for the eigenvalue problem (63) in the coordinate representation. The coordinate representation for the scaling fields reads

$$
W(\phi, z) = \sum_{i \in \mathscr{I}} \mathscr{O}_i(\phi) \, W^i(z) \tag{77}
$$

In the coordinate representation, the eigenvalue problem (63) becomes a set of algebraic equations

$$
(\sigma_k - \lambda(z)) \, W^k(z) = 2z \sum_{i, j \in \mathscr{I}} F^k_{i, j} V^i(z) \, W^j(z) \tag{78}
$$

It can be solved recursively in an interpolation expansion

$$
W^k(z) = \sum_{r=0}^{\infty} z^r W^{k, r} \tag{79}
$$

The power series (35), (64), and (79) yield a solution to (78) provided that the coefficients satisfy

$$
(\sigma_k - \lambda^{(0)}) \, W^{k, r} - \lambda^{(r)} W^{k, 0}
$$

$$
= \sum_{s=1}^{r-1} \lambda^{(s)} W^{k, r-s} + 2 \sum_{s=0}^{r-1} \sum_{i, j \in \mathscr{I}} F^k i, j V^{i, s} W^{j, r-1-s} \tag{80}
$$

As in the case of the interaction, the system of Eqs. (80) can be organized into a recursion relation. To order zero (80) reads

$$(\sigma_k - \lambda^{(0)}) \, W^{k, \, 0} = 0 \tag{81}$$

It tells us that we should select one of the $k \in \mathcal{S}$ as zeroth order eigenvector. We choose $\underline{1} = (1, 0)$ for definiteness. Then the zeroth order is

$$W^{k, \, 0} = \delta_{\underline{1}, \, k}, \, \lambda^{(0)} = \sigma_{\underline{1}} = 2 \tag{82}$$

The only $k$ with $\sigma_k = 2$ is $k = \underline{1}$. We will again restrict our attention to this nondegenerate case. The below recursion relation is valid for general non-degenerate perturbations, with minor notational changes. The first order equation in the system (80) is given by

$$(\sigma_k - \lambda^{(0)}) \, W^{k, \, 1} - \lambda^{(1)} W^{k, \, 0} = 2F^k_{\underline{2}, \, \underline{1}} \, V^{2, \, 0} \tag{83}$$

Therefrom it follows that the first order correction to the eigenvalue is in the coordinate representation

$$\lambda^{(1)} = -2F^{\underline{1}}_{\underline{2}, \, \underline{1}} \, V^{2, \, 0} \tag{84}$$

We remark that in the degenerate case, the degeneracy is typically lifted by the first order correction to the eigenvalue. The other coefficients to first order are

$$W^{\underline{1}, \, 1} = 0 \tag{85}$$

and, for $k \in \mathcal{S} \backslash \{\underline{1}\}$,

$$W^{k, \, 1} = \frac{2F^k_{\underline{2}, \, \underline{1}}}{\sigma_k - \lambda^{(0)}} \tag{86}$$

This computation generalizes immediately to higher orders. The formula for the order $r$ eigenvalue in terms of lower order data is

$$\lambda^{(r)} = -2 \sum_{s=0}^{r-1} \sum_{i, \, j \in \mathcal{S}} F^{\underline{1}}_{i, \, j} V^{i, \, s} W^{j, \, r-1-s} \tag{87}$$

The order $r$ eigenvector is then given by

$$W^{\underline{1}, \, r} = 0 \tag{88}$$

for $r \geqslant 1$, together with

$$W^{k,r} = \frac{1}{\sigma_k - \lambda^{(0)}} \left\{ \sum_{s=1}^{r-1} \lambda^{(s)} W^{k,r-s} + 2 \sum_{s=0}^{r-1} \sum_{i,j \in \mathscr{I}} F_{i,j}^k V^{i,s} W^{j,r-1-s} \right\} \quad (89)$$

for $k \in \mathscr{I} \backslash \{\underline{1}\}$. Eq. (87), (88), and (89) define a recursive perturbation expansion for the critical indices of the nontrivial fixed point.

## 8. COMPUTATION OF THE RECURSIONS

We computed the $z$-expansion for the potential recursively by means of (42) and (43), and for the eigenvalue problem by means of (87) and (89) using computer algebra. We restricted our attention to the case of three dimensions. It turned out to be crucial to compute the structure coefficients to high accuracy. We calculated them to an accuracy of 45 digits with Maple V. The perturbation expansion was performed up to a maximal order of 25. The derivative expansion was performed up to $\alpha_{max} = 20$ orders of $p^2$ in the 2-point vertex. Table 1 shows the series for the $\phi^4$-coupling both in the ultra-local approximation $\alpha_{max} = 0$ and for $\alpha_{max} = 4$ up to the order $z^{11}$. The coefficients prove to increase in absolute value proportional to $C^n n!$ with some constant $C$. Their signs alternate. From this behavior we conclude that the series does not converge but is Borel summable. A proof of local Borel summability will be presented elsewhere. The constant $C$ is related to an instanton singularity of the Borel transform on the negative

Table 1. Examples for the Behavior of the Expansion Coefficients

| $n$ | $V_{2,0}^{(n)}, \alpha_{max} = 0$ | $V_{2,0}^{(n)}, \alpha_{max} = 4$ |
|---|---|---|
| 0 | $2.1874 \cdot 10^{-1}$ | $2.1874 \cdot 10^{-1}$ |
| 1 | $4.5814 \cdot 10^{-1}$ | $4.5814 \cdot 10^{-1}$ |
| 2 | $-8.7171 \cdot 10^{-1}$ | $-8.6761 \cdot 10^{-1}$ |
| 3 | $4.6575 \cdot 10^{0}$ | $4.6815 \cdot 10^{0}$ |
| 4 | $-4.0553 \cdot 10^{1}$ | $-4.0546 \cdot 10^{1}$ |
| 5 | $4.2980 \cdot 10^{2}$ | $4.2992 \cdot 10^{2}$ |
| 6 | $-5.2117 \cdot 10^{3}$ | $-5.2130 \cdot 10^{3}$ |
| 7 | $7.0118 \cdot 10^{4}$ | $7.0133 \cdot 10^{4}$ |
| 8 | $-1.0267 \cdot 10^{6}$ | $-1.0269 \cdot 10^{6}$ |
| 9 | $1.6155 \cdot 10^{7}$ | $1.6158 \cdot 10^{7}$ |
| 10 | $-2.7080 \cdot 10^{8}$ | $-2.7084 \cdot 10^{8}$ |
| 11 | $4.8059 \cdot 10^{9}$ | $4.8066 \cdot 10^{9}$ |

**Table 2.  Examples for the Behavior of the Mass Coefficients
at Order $z^{10}$ for $\alpha_{max} = 5$ and $\alpha_{max} = 10$**

| $\alpha$ | $V_{2,\alpha}^{(10)}$, $\alpha_{max} = 5$ | $V_{2,\alpha}^{(10)}$, $\alpha_{max} = 10$ |
|---|---|---|
| 0 | $-1.08730107 \cdot 10^5$ | $-1.-8730280 \cdot 10^5$ |
| 1 | $-7.30254527 \cdot 10^4$ | $-7.30254464 \cdot 10^4$ |
| 2 | $9.42673139 \cdot 10^3$ | $9.42673053 \cdot 10^3$ |
| 3 | $-1.47739400 \cdot 10^3$ | $-1.47739385 \cdot 10^3$ |
| 4 | $2.14057814 \cdot 10^2$ | $2.14057791 \cdot 10^2$ |
| 5 | $-2.79810089 \cdot 10^1$ | $-2.79810056 \cdot 10^1$ |
| 6 | $0.00000000 \cdot 10^0$ | $3.33201984 \cdot 10^0$ |
| 7 | $0.00000000 \cdot 10^0$ | $-3.70314164 \cdot 10^{-1}$ |
| 8 | $0.00000000 \cdot 10^0$ | $3.99772841 \cdot 10^{-2}$ |
| 9 | $0.00000000 \cdot 10^0$ | $-4.43094961 \cdot 10^{-3}$ |
| 10 | $0.00000000 \cdot 10^0$ | $5.30798357 \cdot 10^{-4}$ |

real axis. It can be seen as an accumulation point of poles when the series
is converted into various Pade approximants. The derivative expansion on
the other hand proves to converge. This is illustrated in Table 2 for two
values of $\alpha_{max}$. We note in passing that the difference between $\alpha_{max} = 5$ and
$\alpha_{max} = 10$ is small.

The spectrum of the nontrivial fixed is computed along the strategy
explained in Section 6 and 7. It requires as an input the fixed point interac-
tion in $z$-expansion. We evaluated it for all values of $\alpha_{max}$ inbetween zero
and twenty. In the following we will concentrate on an estimate of the
critical index $v$ (68) by resummation of the series for all these twenty one
approximations. We computed the series by means of (87) and (89) to
order twenty five of $z$-expansion.

Table 3 shows as an example the series for the eigenvalue $\lambda$ in the
ultra-local case $\alpha_{max} = 0$ and in the case of $\alpha_{max} = 4$ up to the order twelve
of perturbation theory. Again the series alternate, and the coefficients grow
in absolute value as $C^n n!$. The series are therefore not expected to converge.
We remark that a proof thereof is however missing. The Borel transform
of a series with this asymptotics has a finite radius of analyticity $R_{\alpha max}$. It
is determined by an instanton singularity on the negative real axes of the
complex Borel plane. This radius of analyticity is an interesting quantity. It
can be investigated by a number of methods, see [DI89] and references
therein. One of them is the Padé method. Recall that the Padé approxi-
mant of order $(l, m)$ for a function $f$ is a rational function $f_{l,m}(z) = P_l(z)/
Q_m(z)$. Here $P_l$ and $Q_m$ are polynomials of degree $l$ and $m$ respectively,
determined such that the taylor expansions of $f$ and $f_{l,m}$ agree up to order
$z^{l+m}$. One then observes that the poles of the various possible Padé

**Table 3.  Series Coefficients for $\lambda$ up to Order 12 for**
**$\alpha_{max} = 0$ and $\alpha_{max} = 4$**

| $n$ | $\lambda^n$, $\alpha_{max} = 0$ | $\lambda^n$, $\alpha_{max} = 4$ |
|---|---|---|
| 0 | $2.000000 \cdot 10^0$ | $2.000000 \cdot 10^0$ |
| 1 | $-3.333333 \cdot 10^{-1}$ | $-3.333333 \cdot 10^{-1}$ |
| 2 | $-3.490659 \cdot 10^{-1}$ | $-3.490659 \cdot 10^{-1}$ |
| 3 | $1.148993 \cdot 10^0$ | $1.159189 \cdot 10^0$ |
| 4 | $-7.414413 \cdot 10^0$ | $-7.369227 \cdot 10^0$ |
| 5 | $6.358855 \cdot 10^{-1}$ | $6.358630 \cdot 10^1$ |
| 6 | $-6.649232 \cdot 10^2$ | $-6.646081 \cdot 10^2$ |
| 7 | $7.999490 \cdot 10^3$ | $7.996744 \cdot 10^3$ |
| 8 | $-1.070838 \cdot 10^5$ | $-1.070532 \cdot 10^5$ |
| 9 | $1.562548 \cdot 10^6$ | $1.562166 \cdot 10^6$ |
| 10 | $-2.452524 \cdot 10^7$ | $-2.452002 \cdot 10^7$ |
| 11 | $4.103373 \cdot 10^8$ | $4.102601 \cdot 10^8$ |
| 12 | $-7.271917 \cdot 10^9$ | $-7.270698 \cdot 10^9$ |

approximants accumulate around a cut or a singularity of $f$. With the Padé method we found

$$R_{\alpha\,max} = 0.88 \pm 0.02 \tag{90}$$

with no significant dependence of $\alpha_{max}$. Figure 1 shows a plot of all poles of all Padé approximants $(B\lambda)_{l,m}$ with $l+m = 25$ in the complex Borel plane for the two cases $\alpha_{max} = 0$ and $\alpha_{max} = 4$ respectively. Here $B\lambda$ denotes
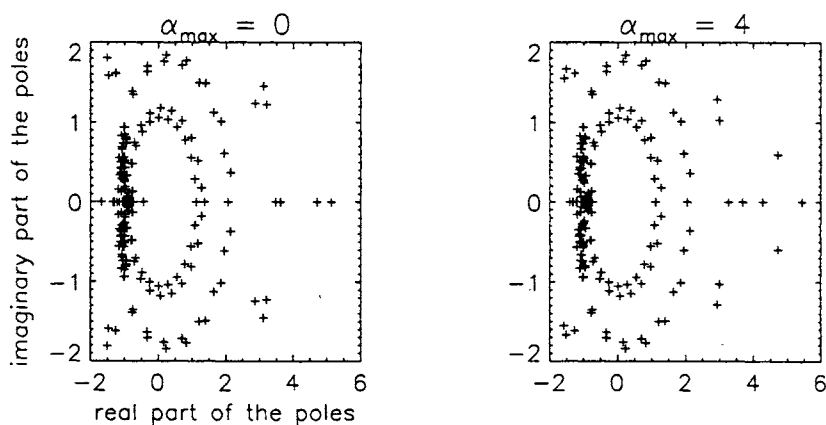


Fig. 1.   Radius of convergence of $B\lambda$ by the Padé method for $\alpha_{max} = 0$ and $\alpha_{max} = 4$.

the Borel transform of $\lambda$. As expected, the poles accumulate on the negative real axes. Notice however that there are many spurious singularities on and nearby the positive real axes. These spurious poles endanger the inverse Borel transform as a contour integral along the positive real axis. The pictures for $\alpha_{max} = 0$ and $\alpha_{max} = 4$ show tiny differences. For instance, the poles on the positive real axis are not on fixed locations and can therefore be regarded as spurious.

## 9. DETERMINATION OF v

To compute the value of the critical index $v$, we have to evaluate the $z$-expansion at $z = 1$. Naive evaluation does not give a meaningful answer since the expansion does not converge. Therefore we had to rely on resummation technology. A review of series resummation and references to the original literature is given in [ZJ89] and [DI89]. We tried four standard methods and compared the results.

First we computed (for all values of $\alpha_{max}$ between zero and twenty) all Padé·approximants $(\lambda)_{l,m}$ with $l + m \leqslant 25$, and evaluated them at $z = 1$. These values are conveniently displayed in a Padé table ($l$-$m$ grid). To get an idea for the value of $\lambda$ and an estimate for the error we computed the mean value and deviation for the lines of fixed order in $z(l + m = \text{const})$ in these diagrams after having discarded all values below a lower value $\lambda_{min}$ and above an upper value $\lambda_{max}$. The idea thereof is that large deviations come from spurious singularities. We were careful not to choose the window too narrow. Our error estimate should be regarded as rather pessimistic. If these mean values converge with increasing order in $z$ we use them and an inspection of the whole table to find an estimate for the value of $v$.

In the second method ($D$ log) one computes the Padé approximants for the logarithmic derivative $\lambda'(z)/\lambda(z)$. $\lambda$ is then reconstructed as the exponential of an integral

$$\lambda_{l,m} = \lambda(0)\, e^{\int_0^1 dz((d/dz)\,\log \lambda(z))_{l,m}} \tag{91}$$

The integration can be performed numerically to high accuracy. The $D$ log method is particularly efficient when the singularity is of the type $\lambda(z) = A/(x - x_c)^\gamma$ with a nonintegral exponent $\gamma$.

The third proposal is to use a Padé approximants for the Borel transform of the series. The Borel transform of a power series $f(x) = \sum_{n \geqslant 0} f_n x^n$ is defined by

$$(Bf)(z) = \sum_{n \geqslant 0} \frac{f_n}{n!} z^n \tag{92}$$

The Borel transform of power series with finite radius of convergence defines an analytic continuation of the function to a maximal simplex through the integral

$$f(x) = \int_0^\infty dt \, e^{-t} (Bf)(xt) \tag{93}$$

Again we get Padé tables of approximants for $\lambda$ by numerically integrating this back transformation for various Padé approximants of $B\lambda$.

The off diagonal estimates in these tables can be improved by using information on the analyticity properties of the Borel transform. $Bf$ could for instance have a cut along $(-\infty, -R]$ on the negative real axes. Let us assume that this is indeed the case (with $R = 0.88 \pm 0.02$). Then the cut plane can be mapped conformally via

$$u(z) = \frac{\sqrt{z/R+1}-1}{\sqrt{z/R+1}+1}; \qquad z = \frac{4Ru}{(1-u)^2} \tag{94}$$

onto the unit circle. Under this mapping $(Bf)(z)$ transforms to $(\tilde{B}f)(u)$. We then use Padé approximants for the mapped series. A Padé table for $\lambda$ is obtained via the inverse transformation

$$\lambda_{l,m} = 4R \int_0^1 du \, \frac{1+u}{(1-u)^3} e^{-4Ru/(1-u)^2} (\tilde{B}f)_{l,m}(u) \tag{95}$$

The outcome of this method relieves on a careful estimate of the radius of convergence of the Borel transform. For each $\alpha_{\text{max}}$, we calculated three estimates for $\lambda$, one for our estimated value of $R$ and one for $R + \Delta R$ and $R - \Delta R$ respectively, where $\Delta R$ means the error in our estimate for the error of the radius. The inspection of all three Padé tables yields $\lambda$ and an error estimate.

We also tried out inhomogenous differential approximants, but we could see no improvement as compared with Padé or $D \log$ Padé approximants. The integration of the differential equations in this method turned out to be both time consuming and fragile due to the poles close to the origin.

In Table 4 we summarize our results for $v$ for the different values of $\alpha_{\text{max}}$. The errors refer as usually to the last digit. We come to the following conclusions. The Padé method is the least precise one with an error of about 0.01. From it we can get an idea about the value of $v$, but no accurate estimate. The errors of the $D \log$ Padé method ($D \log$ ) and the Borel Padé method (BP) are of comparable size. The Borel Padé method with conformal mapping (BPconf) has the least errorbars. At higher orders

Table 4.   Results for the Critical Exponent ν with the Padé Method, the D log
Padé Method (D log), the Borel Padé Method (BP), and the Borel Padé
Method with Conformal Mapping (BPconf) the Various Orders of the
Derivative Expansion of the 2-Point Vertex

| $\alpha_{max}$ | ν Padé | ν D log | ν BP | ν BPconf |
|---|---|---|---|---|
| 0 | 0.6630(20) | 0.6640(10) | 0.6599(30) | 0.6630(30) |
| 1 | 0.6200(150) | 0.6180(50) | 0.6150(70) | 0.6100(10) |
| 2 | 0.6340(130) | 0.6290(40) | 0.6264(18) | 0.6300(10) |
| 3 | 0.6300(110) | 0.6220(40) | 0.6220(70) | 0.6200(40) |
| 4 | 0.6320(90) | 0.6260(30) | 0.6286(68) | 0.6260(10) |
| 5 | 0.6300(100) | 0.6260(40) | 0.6240(50) | 0.6256(10) |
| 6 | 0.6330(100) | 0.6260(40) | 0.6290(40) | 0.6270(10) |
| 7 | 0.6310(110) | 0.6280(70) | 0.6220(60) | 0.6260(20) |
| 8 | 0.6330(90) | 0.6230(70) | 0.6310(60) | 0.6266(4) |
| 9 | 0.6290(90) | 0.6260(80) | 0.6230(60) | 0.6220(40) |
| 10 | 0.6330(90) | 0.6260(40) | 0.6320(70) | 0.6286(3) |
| 11 | 0.6300(110) | 0.6260(50) | 0.6200(150) | 0.6310(50) |
| 12 | 0.6310(150) | 0.6260(60) | 0.6340(60) | 0.6305(25) |
| 13 | 0.6280(120) | 0.6260(80) | 0.6200(200) | 0.6350(60) |
| 14 | 0.6360(110) | 0.6270(50) | 0.6410(80) | 0.6440(30) |
| 15 | 0.6330(100) | 0.6250(60) | 0.6200(180) | 0.6440(30) |
| 16 | 0.6340(200) | 0.6310(40) | 0.6520(50) | 0.6300(50) |
| 17 | 0.6250(150) | 0.6270(200) | 0.6320(200) | 0.6259(40) |
| 18 | 0.6380(60) | 0.6380(110) | 0.6549(58) | 0.6590(160) |
| 19 | 0.6420(150) | 0.6000(400) | 0.6060(130) | 0.6240(70) |
| 20 | 0.6420(80) | 0.6420(200) | 0.6550(40) | 0.6430(160) |

of the derivative expansion of the 2-point vertex the errors increase
significantly. To display this effect, we have plotted the data of Table 4 in
Fig. 2. One can see that the values for ν oscillate around a mean value.
Up to a certain order this sequence seems to converge. Thereafter, the
difference between $\nu(\alpha_{max})$ and $\nu(\alpha_{max+}1)$ and the error grows.

We believe this effect to be the consequence of a numerical instability.
In high orders of derivative expansion and high orders of perturbation
theory one is dealing with numbers of enormously varying magnitudes (in
our case one hundred orders). In practice we computed our series to an
accuracy of 45 digits, and a problem arises in the cancellation of large
numbers in the course of the recursion. A more destructive explanation
would be that the resummation fails to produce a convergent derivative
expansion, or even more desastrous that the non-perturbative kernels are
not analytic functions of the momenta. The final answer to this question
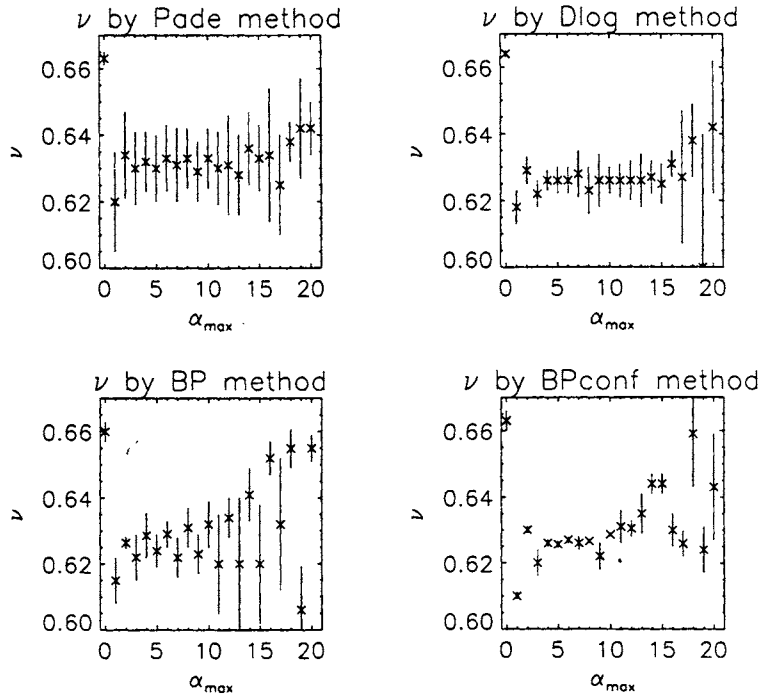can only be given on the basis of a nonperturbative construction of the

Fig. 2. The critical index $v$ as a function of the order of the derivative expansion of the 2-point function $\alpha_{max}$ for the four extrapolation methods.

fixed point and is outside the scope of this paper. Our insight comes from the evaluation of various approximants to different orders of accuracy.

We confine our further discussion to those values of $\alpha_{max}$ which lie before the onset of instability. The $D$ log method and the BPconf method both yield nearly constant values for $v$ at orders between $\alpha_{max} = 4$ and $\alpha_{max} = 12$ and between $\alpha_{max} = 4$ and $\alpha_{max} = 8$ respectively. We propose this value to be the limit of $v$ at arbitrary order of the derivative expansion.

Consider the data for the ultra-local case $\alpha_{max} = 0$ and to first order $\alpha_{max} = 1$ of derivative expansion. For the ultra-local case, which can be compared with the hierarchical model ($v = 0.6501625$, [KW88]) we find $v = 0.6625(33)$ which is bigger than the full critical index. Disregarding the pure Padé estimate, we get for $\alpha_{max} = 1$ the result $v = 0.6144(62)$. This value is considerably lower than the value at $\alpha_{max} = 0$ and even lower than the full critical index. In view of the tiny differences between the fixed point coefficients at $\alpha_{max} = 0$ and $\alpha_{max} \neq 0$, we find this surprising. Compare for example the coefficients in Table 1.

**Table 5.   Results for the Critical Exponent $v$ of the Full Model**

| $v$ | Method | Literature |
|---|---|---|
| 0.6300(15) | three dimensional renormalization group | [GZJ80] |
| 0.6298(7) | | [BB85] |
| 0.630 | | [N91] |
| 0.6305(25) | renormalization group, $\varepsilon$-expansion | [GZJ85] |
| 0.6301 | high temperature series | [R95] |
| 0.6300(15) | high temperature series for bcc-grid | [NR90] |
| 0.6289(8) | Monte-Carlo methods | [FL91] |
| 0.6301(8) | | [BLH95] |
| 0.625(1) | Monte-Carlo renormalization group | [GT96] |
| 0.626(9) | Scaling-field method | [NR84] |

At higher orders of derivative expansion, the values for $v$ oscillate and converge to a mean value. The limit value has been determined as the mean values of $v$ over the nearly constant plateaus. As best estimate for the BPconf method we get $v = 0.6262(13)$. The $D$ log method yields $v = 0.6259(57)$. These results should be compared with the critical index $v$ of the three-dimensional Ising model in the literature. In Table 5 we list a few results for $v$. A comprehensive article on this issue is [BLH95]. It also contains an overview of experimental data. With series expansion and Monte Carlo methods one gets $v = 0.630$. On the other hand the Monte Carlo renormalization group suggests $v = 0.625$. This gap is object of current discussions. Our value is closest to the value of [NR84] and [GT96].

## 10. SUMMARY OF THE $z$-EXPANSION

In this article we investigated a form of Wilsons infinitesimal renormalization group The starting point was Eq. (1). We found a practical way to solve the equation in a systematic manner. The central idea was to introduce an interpolating parameter $z$, which continuously turns on the nonlinear term in (1). Everything was expanded in this parameter. The interpolation was arranged such that the zeroth order is a $\phi^4$-vertex. The expansion was presented both in a coordinate free representation and in coordinate form, where the interaction is expanded in a basis of vertices. As a basis we advocated the use of a full two point interaction in derivative expansion together with local vertices of any power of fields. Derivative interactions of higher powers were neglected. The basis of interactions came encoded in a system of scaling dimensions and structure constants. Their evaluation was reduced to a one-dimensional Feynman integral

which we evaluated numerically. We reformulated our expansion into recursive equations for the fixed point interaction, its scaling fields, and their anomalous dimensions. We performed a detailed analysis of the series for the critical exponent associated with a massive perturbation of the fixed point. The result is a new and independent calculation of the critical index $v$ of the the three-dimensional Ising model. We solved the recursion relations for the eigenvalue problem up to high orders and analyzed the resulting series by means of four different extrapolation methods. Our best estimator for the critical index $v$ is $v = 0.6262(13)$. We compared our results with values for the critical exponent $v$ known in the literature.

The results encourage us to further investigations. On the menu of open problems we have the inclusion of momentum dependent higher vertices for the scalar model, theoretical estimates on the $z$-expansion, and the generalization to vector and matrix models. We hope to return with accurate data on their critical properties by means of $z$-expansion in the near future.

## 11. THE ANOMALOUS DIMENSION $\eta$

We use a renormalization group transformation which rescales the field $\phi$ canonically. The anomalous dimension $\eta$ of the field $\phi$ is consequently zero at our fixed point. In $z$-expansion this can be seen as follows. We computed the scaling field associated with the one point function

$$W(\phi, z) = \int d^3x \, \phi(x) + O(z) \tag{96}$$

Its eigenvalue comes out as the canonical value $1 + D/2 = 5/2$, independent of $z$. All higher orders in $z$ cancel out exactly. This can be shown both in the full $z$-expansion and also in our approximate evaluation. Therefore, $\eta$ is zero to any order of $z$-expansion.

Our understanding of an anomalous dimension is the following. The origin of an anomalous dimension $\eta$, different from zero, is a symmetry of the renormalization group, associated with a global—that is space-time independent—rescaling of the field. In our setup this symmetry is known explicitly. Our fixed point equation reads (zero equals to)

$$R_z[:V:_v](\psi) = \left\{ ([\mathscr{D} - 1 + z]\,\psi, \delta_\psi) + \frac{1}{2}(\delta_\psi, \Gamma\delta_\psi) \right\} :V(\psi):_v$$

$$- \frac{z}{2}(\delta_\psi :V(\psi):_v, \Gamma\delta_\psi :V(\psi):_v) \tag{97}$$

where $\Gamma = 2\chi$ and $v = \chi/(-\Delta)$. The functional operator $R_z$ generates the interpolated renormalization group. The rescaling symmetry is generated by

$$U[\,:V:_v](\psi) = \{(\mathscr{D}'\psi, \delta_\psi) + \tfrac{1}{2}(\delta_\psi, \Gamma'\delta_\psi)\} :V(\psi):_v$$
$$- \tfrac{1}{2}(\delta_\psi :V(\psi):_v, \Gamma'\delta_\psi :V(\psi):_v) + \tfrac{1}{2}(\psi, (-\Delta)\psi) \qquad (98)$$

where $\mathscr{D}' = -1/2 + \chi$ and $\Gamma' = \chi(1-\chi)/(-\Delta)$. It commutes with $R_1$, but it does not commute with $R_z$ for $z < 1$. In other words, our interpolation explicitely breaks the resealing symmetry. A fixed point with anomalous dimension is a solution of

$$R_z[\,:V(\psi):_v](\psi) = \eta U[\,:V(\psi):_v](\psi) \qquad (99)$$

at $z = 1$; in conjunction with a normalization condition on $:V(\psi):_v$ which breaks the rescaling symmetry. Formal power series solutions (in $z$) thereof exist for all values of $\eta$. The question of symmetry restauration at $z = 1$ is thus beyond formal $z$-expansion. We have chosen $\eta = 0$ in this situation. It can be viewed as the zeroth approximation in a double expansion in both $z$ and $\eta$, the latter being possible after interpolation. A standard proposal for how to determine $\eta$ is to demand an eigenvalue zero in the spectrum of the fixed point. This question arises when one computes order $\eta$ corrections to our scheme. We postpone it to subsequent work.

Although it seems unlikely to us, there remains the possibility of a nontrivial infrared fixed point without anomalous dimension. A recent construction of Brydges *et al.* [BDH97] provides an example where this happens.

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